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ADDENDUM

Diffusion properties of test particles in a two-dimensional fluid

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Abstract. We study by means of numerical simulation, the diffusive behaviour of test particles in a two-dimensional fluid, whose velocity field passes from periodic to chaotic motion. We show that there exists no relation between the diffusion coefficients and the degree of chaoticity of the Lagrangian motion. This is related to the intermittent character of the diffusive process; this behaviour, on a qualitative ground, may be interpreted in terms of intermittent maps.

Recently we have analysed the properties of the motion of particles in a two-dimensional fluid [1]. Especially we studied the relation of the Lagrangian chaoticity to the Eulerian properties of the flow. In this addendum we turn our attention to the diffusion properties of test particles plunged in the velocity field of [1]. Indeed in many circumstances for the diffusion and mixing properties of a system the 'convective' diffusion, linked to Lagrangian features, is more important than the 'molecular' diffusion.

Let us briefly review the model and the main results of [1]. Following a standard procedure in the study of weak turbulence, we take Navier–Stokes equations and, by Fourier transforming the stream function, we get a (infinite) system of ordinary differential equations, from which we select a suitable (finite) set [2]. We are left with the following dynamical system:

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{f}_{Re}(\boldsymbol{\gamma}) \qquad \boldsymbol{\gamma}, \boldsymbol{f} \in R^5 \tag{1a}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\gamma}(t)) \qquad \boldsymbol{x} \in R^2 \tag{1b}$$

where the amplitudes of the Fourier series, $\boldsymbol{\gamma}$, are the variables describing the Eulerian velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, Re is the Reynold's number and $\boldsymbol{x} = (x_1, x_2)$ is the particle position. The velocity field is periodic in the two spatial dimensions with wavelength 2π. Equation (1b) may describe the behaviour both of a fluid particle and of a test particle, that is small enough not to disturb the velocity field but also big enough not to perform a Brownian motion. Let us remark that the incompressibility condition $\boldsymbol{u} = (\partial_2\psi, -\partial_1\psi)$, makes (1$b$) a time-dependent Hamiltonian system for which the stream function ψ has the role of the Hamiltonian function.

System (1a) shows the following behaviour: if $Re < Re_1 = 22.8538\ldots$ there are stable fixed points which, for $Re = Re_1$, lose their stability while, via Hopf bifurcation, stable periodic orbits appear; these become unstable and bifurcate to double-period orbits for $Re = Re_2 = 28.41\ldots$; and so on along a birfurcation cascade that leads to chaos for $Re > Re_c \simeq 28.73$.

The Lagrangian part (i.e. equation (1*b*)) behaves as follows: when $Re < Re_1$ the motion is regular (either periodic or open depending on the initial conditions); when $Re = Re_1 + \varepsilon$ and $\varepsilon < \varepsilon_c \simeq 0.7$, chaotic layers appear around the separatrices (these latter are shown in figure 1), but far from them the motion is still regular, as before; finally if $\varepsilon > \varepsilon_c$ the chaotic layers touch each other and the particle diffuses away independently of the initial position. We note that there are two kinds of separatrices: the isolated 'eights', labelled by A, and the periodic ones, labelled by B in figure 1. In [1] we have shown that Lagrangian chaos has a connection with properties of the Eulerian field only when the Eulerian part (i.e. equation (1*a*)) exhibits a Hopf bifurcation; the onset of Eulerian chaos has no influence on the Lagrangian properties.

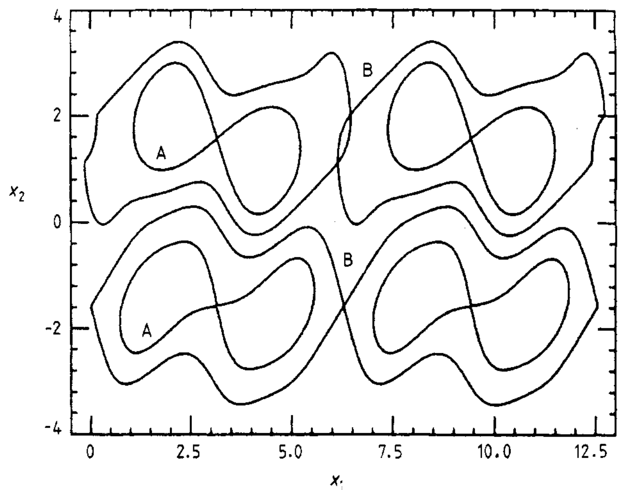

Figure 1. Structure of the separatrices for equation (1*b*) as given by the 5-modes truncated model of [2].

In this addendum we discuss diffusion for $\varepsilon > \varepsilon_c$ (smaller values of ε give rise to ballistic or limited motions, depending on the initial conditions, besides the diffusive ones). We study the behaviour of direction 1 and 2 diffusion coefficients, D_1 and D_2, on variation of ε, and compare them with other quantities such as: mean quadratic particle velocity, $\overline{v^2}$, maximum Lyapunov characteristic exponent, λ [3]. Let us define

$$D_1 = \frac{1}{2} \lim_{T\to\infty} \frac{1}{T} \overline{(x_1(t+\tau) - x_1(t))^2} \tag{2}$$

$$D_2 = \frac{1}{2} \lim_{T\to\infty} \frac{1}{T} \overline{(x_2(t+\tau) - x_2(t))^2} \tag{3}$$

$$\overline{v^2} = \overline{|\dot{x}(t)|^2} \tag{4}$$

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|z(t)|}{|z(0)|} \tag{5}$$

where $z(t)$ is the tangent vector given by the linearised equation

$$\dot{z}_i = \frac{\partial u_i}{\partial x_1} z_1 + \frac{\partial u_i}{\partial x_2} z_2 \qquad i = 1, 2. \tag{6}$$

The temporal mean $\overline{(\cdot)}$ is taken over a very long time.

A natural question to put is what is the relation of D_1 and D_2 with λ and $\overline{v^2}$; a hasty answer, based on dimensional arguments and analogy with the Brownian motion, is

$$D_{1,2} \sim \lambda. \tag{7}$$

The line of reasoning goes as follows. One assumes for the uncorrelation time that $\tau_u \sim 1/\lambda$; then one has $D_{1,2} \sim L^2/\tau_u$ and therefore equation (7), assuming that L (the typical distance the particle runs along during the time τ_u) does not depend on *Re*. We note that equation (7) has been observed to occur for the diffusion of charged particles in a turbulent plasma, which is a problem equivalent to plane hydrodynamics [4].

Let us now recall that the diffusion coefficient D_i is related, via the Kubo formula, to the velocity correlation function

$$D_i = \int_0^\infty \overline{v_i(t+\tau)v_i(t)}\, \mathrm{d}\tau \sim \overline{v^2}\tau_d \tag{8}$$

where τ_d is the typical time for the decay of the correlation, so that, making the likely choice $\tau_d \sim 1/\lambda$, one gets the other possibility

$$D_{1,2} \sim \overline{v^2}/\lambda. \tag{9}$$

We show the behaviour of D_1, D_2, λ, $\overline{v^2}$ as functions of $\mathrm{Re} - \mathrm{Re}_1$ in figure 2. It is apparent that D_1 initially decreases and then is nearly constant while ε increases, but D_2 is an increasing function of ε.

It seems that no simple relation (like equations (7) or (9)) exists between D_i and λ. We have that the above-mentioned statistical arguments are completely wrong even on a qualitative level. The main source of this strong disagreement is due to the fact that for $\varepsilon \sim O(1)$ the autocorrelation function of $\dot{x}_1 = v_1$ decays on times very much larger than $1/\lambda$. We stress that the horizontal diffusion for ε slightly larger than ε_c is ruled by the intervals of regular motion. There is no likeness with the standard diffusion in the Brownian motion: our case resembles the random walk with a probability distribution for pausing times between successive steps in the walk [5]. Roughly speaking, we have the following scenario: regular ballistic motion for a certain time interval (that on the average diverges as ε goes to ε_c), entrapping inside a little region of disorderly motion, and then ballistic motion again (not necessarily in the same sense as the preceding one).

In figure 3 examples of (x_1, x_2) as functions of time are shown. For $\varepsilon \sim \varepsilon_c$, the intermittent behaviour is evident, as also is the difference with the standard random walk: long horizontal escapes, then entrappings, and so on. D_1 seems to diverge when ε approaches ε_c from above, D_2 appears to go to zero.

We have not been able to determine the exact value of ε_c and the precise quantitative behaviour of the diffusion coefficients, because of the high computation time required; yet the data we obtain suggest an interpretation of the diffusion process (at least on qualitative grounds) in terms of unidimensional intermittent maps. For the horizontal

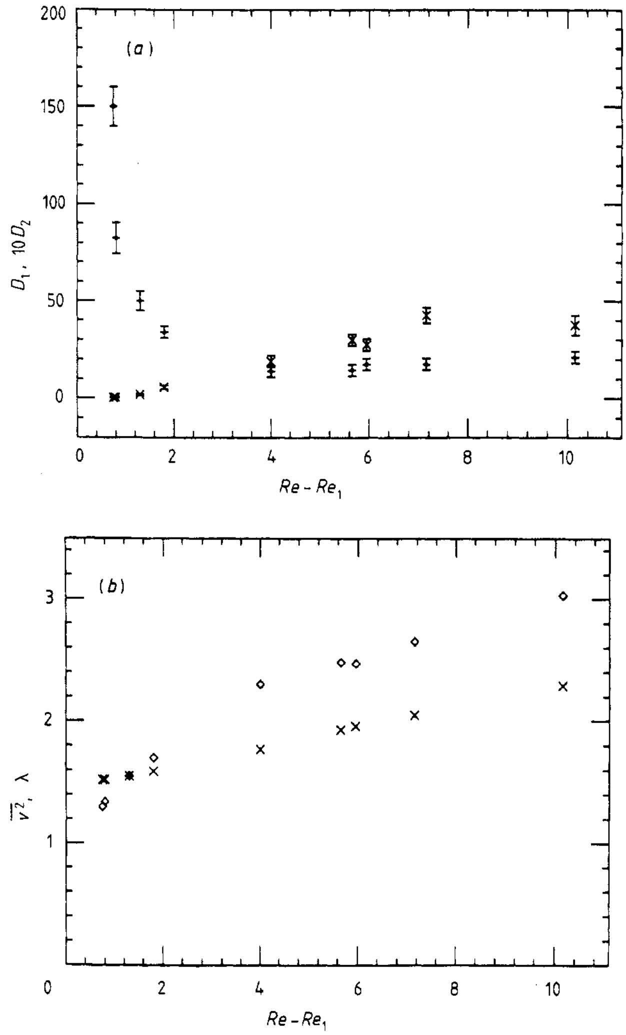

Figure 2. (*a*) Diffusion coefficients D_1 (✱) and $10D_2$ (×) as functions of $Re - Re_1$. (*b*) Mean quadratic particle velocity $\overline{v^2}$ (×) and maximum Lyapunov exponent λ (◇) as functions of $Re - Re_1$.

motion we have the analogy of an intermittent transition from ballistic motion ($\varepsilon < \varepsilon_c$) to a diffusive one; for the motion along the vertical direction the analogy is with a map that makes a transition from a confined motion ($\varepsilon < \varepsilon_c$) to a diffusive one. For maps of the above-mentioned type we have [6], respectively, $D \sim (\varepsilon - \varepsilon_c)^{-1/2}$ and $D \sim (\varepsilon - \varepsilon_c)^{1/2}$, in qualitative agreement with our results. The transition to chaos (tangent contact) of the one-dimensional maps, corresponds in our case to the touching of two chaotic layers of B type, that allows the vertical diffusion. Of course the analogy with the intermittent unidimensional maps cannot be pushed too far because they ought to be interacting.

It is clear that the diffusion properties of this model are rather peculiar and some of them (e.g. divergent D_1) are dependent on the detailed structure of the separatrices, so that we cannot expect that the types of behaviour we met are generic for the

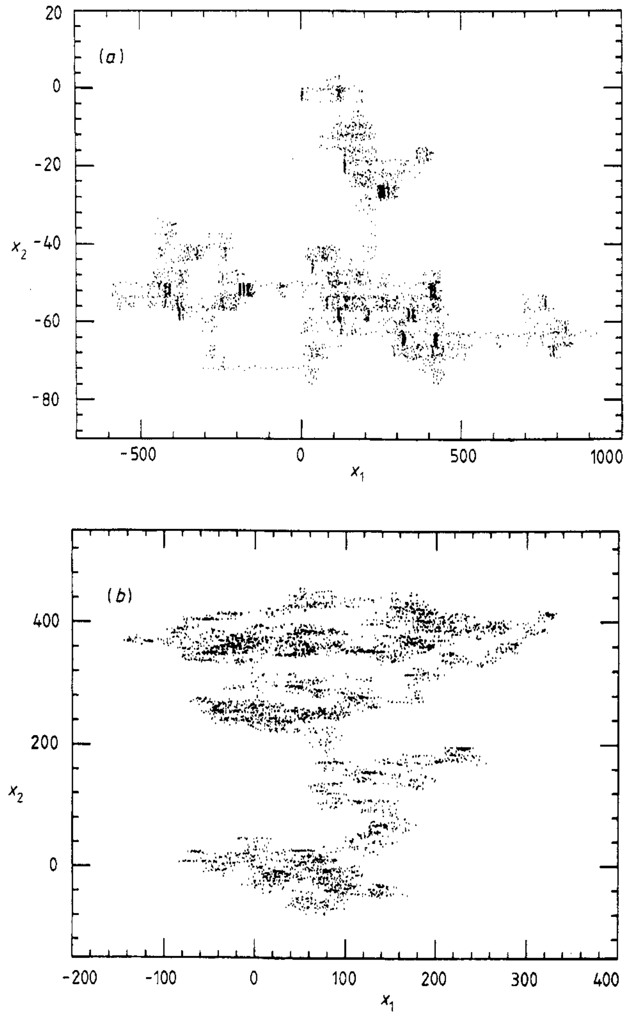

Figure 3. 5000 positions of a particle, driven by equations (1) with (*a*) $Re - Re_1 = 1.3$ and (*b*) $Re - Re_1 = 10.15$, taken every second.

Lagrangian two-dimensional motion. Nevertheless we stress how the high values of D_1 are basically due to the existence of coherent structures which have also been observed in less idealised models of two-dimensional fluids [7].

We conclude by confronting our results with those obtained in different contexts. In [4] Pettini *et al* find a behaviour fitting in with equation (7). In this case, however, the velocity field contains a high number of harmonics (≈ 500), so it is not surprising that the diffusion coefficients change as in equation (7), in agreement with the predictions of the phenomenological theories and statistical approaches. On the other hand Kleva and Drake [8] study the model of [4] with few harmonics, and get a completely different result, even though D still grows with λ. In our case D_1 is more or less constant while λ grows; sometimes it decreases, while D_2 increases with λ.

By comparing our results with these models we may draw the following conclusion: perhaps with the exclusion of limiting cases (i.e. fully developed turbulence) the diffusion coefficients are strongly dependent on the details (coherent structures) and cannot be guessed from statistical arguments.

Finally we mention that the Lagrangian behaviour of Taylor vortices close to the onset of instability [9] also shows diffusion properties similar to ours.

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